ON A PARTICULAR SOLUTION OF THE EULER-POISSON EQUATIONS

(OB ODNOM CHASTNOM RESHENII URAVNENII EILERA-PUASSONA)

PMM Vol.23, No.4, 1959, pp.681-690

E. I. KHARLAMOVA
(Moscow)

(Received 26 June 1958)

In 1947, in his study of regular precessions, Grioli obtained a new particular solution for the problem of the motion of a heavy rigid body about a fixed point [1]. This solution was expressed in article [2] relative to a coordinate system, the axes of which are the principal axes of the inertial ellipsoid for the point of support. It turned out that analytically Grioli's solution is characterized by two particular quadratic integrals of a definite type.

In this article the problem of the existence of new solutions of a similar type is studied. A system of algebraic equations is constructed for the determination of the required parameters. Two solutions of this system are found and investigated. To the first corresponds Grioli's case; the second leads to a new case of integrability [4]. The conditions imposed in this case on the moments of inertia can be fulfilled, if the body has cavities filled with an ideal incompressible fluid [3].

1. Let O be the fixed point of the body, $Oxyz$ coordinate axes fixed in the body, which are the principal axes of the inertial ellipsoid of the body for the fixed point. We shall denote the moments of inertia of the body for these axes by A, B, C. Let the center of gravity lie in one of the principal planes of the inertial ellipsoid, for example in the Oxz plane. The angle between the Ox-axis and the straight line going from the fixed point to the center of gravity of the body will be denoted by a. The product of weight of the body and the distance from the fixed point to the center of gravity will be denoted by l. The variables to be determined are p, q, r, i.e. the projections of the angular velocity of the body on the $Oxyz$ axes, and γ, γ', γ'', i.e. the projections of the unit vector on the same axes, with direction opposite to the direction of the force of gravity. These variables must satisfy the Euler-Poisson equations

$$A\frac{dp}{dt}=(B-C)qr+l\gamma'\sin\alpha,\qquad C\frac{dr}{dt}=(A-B)pq-l\gamma'\cos\alpha$$
$$B\frac{dq}{dt}=(C-A)rp+l\gamma''\cos\alpha-l\gamma\sin\alpha \tag{1.1}$$

$$\frac{d\gamma}{dt}=r\gamma'-q\gamma'',\qquad \frac{d\gamma'}{dt}=p\gamma''-r\gamma,\qquad \frac{d\gamma''}{dt}=q\gamma-p\gamma' \tag{1.2}$$

Three integrals of these equations are known:

$$Ap^2+Bq^2+Cr^2+2l(\gamma\cos\alpha+\gamma''\sin\alpha)=2h \tag{1.3}$$

$$Ap\gamma+Bq\gamma'+Cr\gamma''=m \tag{1.4}$$

$$\gamma^2+\gamma'^2+\gamma''^2=1 \tag{1.5}$$

where h and m are integration constants.

Instead of γ', we will introduce a new variable Γ', and a new independent variable τ, using the relations

$$l\gamma'=q\Gamma',\qquad d\tau=q\,dt$$

Under these conditions equations (1.1) give

$$A\frac{dp}{d\tau}=(B-C)r+\Gamma'\sin\alpha,\qquad C\frac{dr}{d\tau}=(A-B)p-\Gamma'\cos\alpha \tag{1.6}$$

$$\frac{d}{d\tau}\frac{Bq^2}{2}=(C-A)rp+l\gamma''\cos\alpha-l\gamma\sin\alpha \tag{1.7}$$

The equations (1.6) will be linear in p and r if

$$\Gamma'=bp+b''r$$

where b and b'' are constants to be determined. We have

$$A\frac{dp}{d\tau}=(B-C+b''\sin\alpha)r+bp\sin\alpha$$
$$C\frac{dr}{d\tau}=(A-B-b\cos\alpha)p-b''r\cos\alpha \tag{1.8}$$

We will subject the constants b and b'' to the condition

$$\frac{b\sin\alpha}{A-B-b\cos\alpha}=\frac{B-C+b''\sin\alpha}{-b''\cos\alpha}\quad\text{or}\quad\frac{b\cos\alpha}{A-B}-\frac{b''\sin\alpha}{B-C}=1 \tag{1.9}$$

Eliminating with the aid of this equality b'' from the first equation of (1.8), and b from the second, we shall have

$$A\frac{dp}{d\tau}=\frac{b}{A-B}[(A-B)p\sin\alpha+(B-C)r\cos\alpha]$$
$$C\frac{dr}{d\tau}=-\frac{b''}{B-C}[(A-B)p\sin\alpha+(B-C)r\cos\alpha] \tag{1.10}$$

From equations (1.10) we have

$$\frac{Ab''}{B-C}p+\frac{Cb}{A-B}r=n \tag{1.11}$$

From now on we shall assume the integration constant n to be different from zero.

Hence, in the required cases of integrability the relation

$$l\gamma'=q(bp+b''r) \tag{1.12}$$

must hold, where the constants b and b'' are restricted by condition (1.9). These constants have the dimension of the moments of inertia.

The second assumption consists in that q^2 must be a quadratic function of the variables p and r:

$$q^2=\varepsilon_2p^2+\varepsilon_2'pr+\varepsilon_2''r^2+\varepsilon_1p+\varepsilon_1''r+\varepsilon_0 \tag{1.13}$$

Expressions (1.12) and (1.13) are a generalization of the integrals holding in Grioli's solution, whose analytic expression is given in article [2].

The right-hand side of expression (1.13) can be represented with the aid of (1.11), in the form

$$\varepsilon_2p^2+\varepsilon_2'pr+\varepsilon_2''r^2+\frac{1}{n}(\varepsilon_1p+\varepsilon_1''r)\left(\frac{Ab''}{B-C}p+\frac{Cb}{A-B}r\right)+\varepsilon_0$$

Substituting for the product pr its expression in terms of p^2, r^2, n^2 from (1.11), we write the relationship between p, q and r as follows:

$$q^2=\varepsilon p^2+\varepsilon''r^2+e'n^2 \tag{1.14}$$

As a consequence of (1.10) and (1.14), equation (1.7) leads to the following relation:

$$\begin{aligned}l\gamma''\cos\alpha-l\gamma\sin\alpha&=\varepsilon b\frac{B}{A}p^2\sin\alpha-\varepsilon''b''\frac{B}{C}r^2\cos\alpha+\\&+rp\left[\varepsilon b\frac{B}{A}\frac{B-C}{A-B}\cos\alpha-\varepsilon''b''\frac{B}{C}\frac{A-B}{B-C}\sin\alpha-(C-A)\right]\end{aligned} \tag{1.15}$$

We will transform the integral (1.3). The substitution q^2 in the form (1.14) gives

$$Ap^2+Cr^2+B(\varepsilon p^2+\varepsilon''r^2)+2(l\gamma\cos\alpha+l\gamma''\sin\alpha)-2h+Be'n^2=0 \tag{1.16}$$

Instead of h, we will introduce the constant β, using relation

$$Be'n^2 - 2h = \beta n^2 \tag{1.17}$$

Then, taking into account (1.11), equation (1.16) will be written as

$$-l\gamma\cos\alpha - l\gamma''\sin\alpha = \tfrac{1}{2}\left[A + \varepsilon B + \beta\frac{A^2b''^2}{(B-C)^2}\right]p^2 + \\ + \tfrac{1}{2}\left[C + \varepsilon'' B + \beta\frac{C^2b^2}{(A-B)^2}\right]r^2 + \beta\frac{ACbb''}{(A-B)(B-C)}pr \tag{1.18}$$

From equations (1.15) and (1.18) we find

$$l\gamma = ap^2 + a'pr + a''r^2, \quad l\gamma'' = cp^2 + c'pr + c''r^2 \tag{1.19}$$

Here, for brevity, the notations

$$a = -\varepsilon b\frac{B}{A}\sin^2\alpha - \tfrac{1}{2}\left[A + \varepsilon B + \beta\frac{A^2b''^2}{(B-C)^2}\right]\cos\alpha \tag{1.20}$$

$$a' = -\left[\varepsilon b\frac{B(B-C)}{A(A-B)}\cos\alpha - \varepsilon''b''\frac{B(A-B)}{C(B-C)}\sin\alpha - (C-A)\right]\sin\alpha - \\ - \frac{ACbb''}{(A-B)(B-C)}\cos\alpha$$

$$a'' = \varepsilon''b''\frac{B}{C}\cos\alpha\sin\alpha - \tfrac{1}{2}\left[C + \varepsilon''B + \beta\frac{C^2b^2}{(A-B)^2}\right]\cos\alpha$$

$$c = \varepsilon b\frac{B}{A}\sin\alpha\cos\alpha - \tfrac{1}{2}\left[A + \varepsilon B + \beta\frac{A^2b''^2}{(B-C)^2}\right]\sin\alpha$$

$$c' = \left[\varepsilon b\frac{B(B-C)}{A(A-B)}\cos\alpha - \varepsilon''b''\frac{B(A-B)}{C(B-C)}\sin\alpha - (C-A)\right]\cos\alpha - \\ - \frac{\beta ACbb''}{(A-B)(B-C)}\sin\alpha$$

$$c'' = -\varepsilon''b''\frac{B}{C}\cos^2\alpha - \tfrac{1}{2}\left[C + \varepsilon''B + \beta\frac{C^2b^2}{(A-B)^2}\right]\sin\alpha$$

are introduced.

We shall pass to the integral (1.4). Instead of m, we shall introduce the constant μ, using the equality $lm = \mu n^2$. Taking into account formulas (1.19), (1.12) and (1.14), we have

$$Ap(ap^2 + a'pr + a''r^2) + B(\varepsilon p^2 + \varepsilon''r^2)(bp + b''r) + \\ + Cr(cp^2 + c'pr + c''r^2) - \mu n^3 + B(bp + b''r)e'n^2 = 0$$

Substituting for n in the above expression, the left-hand side of the equality (1.11), gives a homogeneous polynomial of the third order in the variables p and r:

$$(aA + \varepsilon Bb)\,p^3 + (a'A + cC + \varepsilon Bb'')\,p^2 r + (c'C + a''A + \varepsilon'' Bb)\,pr^2 +$$
$$+ (c''C + \varepsilon'' Bb'')\,r^3 + e'B\,(bp + b''r)\left(\frac{Ab''}{B-C}\,p + \frac{Cb}{A-B}\,r\right)^2 -$$
$$- \mu\left(\frac{Ab''}{B-C}\,p + \frac{Cb}{A-B}\,r\right)^3 = 0 \qquad (1.21)$$

Hence the variables p and r are related by the expressions (1.11) and (1.21). Since $n \neq 0$, at least one of the coefficients in the left-hand side of (1.11) is not zero. Let it be the coefficient of p. Then expressing p in terms of r, and substituting into (1.21), an equation of the third order is obtained, which in the general case gives a constant value for r. To avoid this case, the requirement must be made that this equality be an identity in r, from which will follow the vanishing of the coefficients of the unknowns in the polynomial (1.21). Thus we shall have, the equations

$$aA + \varepsilon Bb + e'Bb\,\frac{A^2 b''^2}{(B-C)^2} - \mu\,\frac{A^3 b''^3}{(B-C)^3} = 0 \qquad (1.22)$$

$$c''C + \varepsilon'' Bb'' + e'Bb''\,\frac{C^2 b^2}{(A-B)^2} - \mu\,\frac{C^3 b^3}{(A-B)^3} = 0$$

$$a'A + cC + \varepsilon Bb'' + e'B\left[\frac{A^2 b''^3}{(B-C)^2} + 2b\,\frac{ACbb''}{(B-C)\,(A-B)}\right] -$$
$$- 3\mu\,\frac{A^2 C b''^2 b}{(B-C)^2\,(A-B)} = 0 \qquad (1.23)$$

$$c'C + a''A + \varepsilon'' Bb + e'B\left[\frac{C^2 b^3}{(A-B)^2} + 2b''\,\frac{ACbb''}{(B-C)\,(A-B)}\right] -$$
$$- 3\mu\,\frac{C^2 A b^2 b''}{(A-B)^2\,(B-C)} = 0$$

Substituting a and c'' from (1.20) in (1.22), we have two equations for the determination of μ and β:

$$\frac{\beta}{2}\cos\alpha + \mu\,\frac{b''}{B-C} = e'B\,\frac{b}{A} + \left[\left(b\cos\alpha - \frac{A}{2}\right)\varepsilon B - \frac{A^2}{2}\right]\frac{(B-C)^2}{A^3 b''^2}\cos\alpha$$
$$\frac{\beta}{2}\sin\alpha + \mu\,\frac{b}{A-B} = e'B\,\frac{b''}{C} + \left[\left(b''\sin\alpha - \frac{C}{2}\right)\varepsilon'' B - \frac{C^2}{2}\right]\frac{(A-B)^2}{A^3 b^2}\sin\alpha$$

The determinant of these linear equations in $1/2\beta$ and μ is equal, by virtue of (1.9), to unity. Therefore

$$\frac{\beta}{2} = e'B\left[\frac{b^2}{A\,(A-B)} - \frac{b''^2}{C\,(B-C)}\right] + \left[\left(b\cos\alpha - \frac{A}{2}\right)\varepsilon B - \frac{A^2}{2}\right]\frac{(B-C)^2\,b\cos\alpha}{A^3\,(A-B)\,b''^2} -$$
$$- \left[\left(b''\sin\alpha - \frac{C}{2}\right)\varepsilon'' B - \frac{C^2}{2}\right]\frac{(A-B)^2\,b''\sin\alpha}{C^3\,(B-C)\,b^2} \qquad (1.24)$$

$$\mu = e'B\left(\frac{b}{C}\cos\alpha - \frac{b}{A}\sin\alpha\right) -$$
$$-\left[\left(b\cos\alpha - \frac{A}{2}\right)\varepsilon B - \frac{A^2}{2}\right]\frac{(B-C)^2\sin\alpha\cos\alpha}{A^3b''^2} +$$
$$+\left[\left(b''\sin\alpha - \frac{C}{2}\right)\varepsilon'' B - \frac{C^2}{2}\right]\frac{(A-B)^2\sin\alpha\cos\alpha}{C^3b^2}$$

Substituting into (1.23) a', c', c, a'' from (1.20), and β, μ from (1.24), we will have

$$E\varepsilon B + F\varepsilon'' B + G = 0, \qquad E''\varepsilon'' B + F''\varepsilon B + G'' = 0$$
$$E = b'' - \frac{C}{2}\sin\alpha + b\left(\frac{C}{A} - \frac{B-C}{A-B}\right)\sin\alpha\cos\alpha +$$
$$+ C\frac{B-C}{A-B}\frac{b}{b''}\left(1 - 2\frac{b}{A}\cos\alpha\right)\cos\alpha$$
$$E'' = b - \frac{A}{2}\cos\alpha + b''\left(\frac{A}{C} - \frac{A-B}{B-C}\right)\sin\alpha\cos\alpha +$$
$$+ A\frac{A-B}{B-C}\frac{b''}{b}\left(1 - 2\frac{b''}{C}\sin\alpha\right)\sin\alpha$$
$$F = b''\frac{A(A-B)}{C(B-C)}\sin\alpha\left[\sin\alpha + \frac{1}{2}\frac{A(A-B)}{(B-C)}\frac{b''}{b^2}\left(1 - \frac{2}{C}b''\sin\alpha\right)\right]$$
$$F'' = b\frac{C(B-C)}{A(A-B)}\cos\alpha\left[\cos\alpha + \frac{1}{2}\frac{C(B-C)}{(A-B)}\frac{b}{b''^2}\left(1 - \frac{2}{A}b\cos\alpha\right)\right]$$
$$G = \frac{A}{2}(C-2A)\sin\alpha + \frac{A^2(A-B)^2b''^2}{2(B-C)^2b^2}\sin\alpha + AC\frac{B-C}{A-B}\frac{b}{b''}\cos\alpha$$
$$G'' = \frac{C}{2}(A-2C)\cos\alpha + \frac{C^2(B-C)^2b^2}{2(A-B)^2b''^2}\cos\alpha + AC\frac{A-B}{B-C}\frac{b''}{b}\sin\alpha \tag{1.25}$$

Equations (1.25) are used to determine ϵB and $\epsilon'' B$, and depend on A, B, C, b, b'', α. They are independent of e'.

We shall transform the integral (1.5). Substituting (1.19), (1.12), (1.14), (1.11), we will have

$$(ap^2 + a'pr + a''r^2)^2 + (cp^2 + c'pr + c''r^2)^2 + (bp + b''r)^2\left[\varepsilon p^2 +\right.$$
$$\left. + \varepsilon'' r^2 + e'\left(\frac{Ab''}{B-C}p + \frac{Cb}{A-B}r\right)^2\right] = \frac{l^2}{n^4}\left(\frac{Ab''}{B-C}p + \frac{Cb}{A-B}r\right)^4 \tag{1.26}$$

By the same reasoning as above regarding the polynomial (1.21), the homogeneous polynomial (1.26) is an identity in p and r. Therefore, its coefficients must be equal to zero. Taking into account expressions (1.20), we will have the following five equations:

$$\varepsilon^2 B^2 \frac{b^2}{A^2} \sin^2 \alpha + b^2 \left[\varepsilon + e' \frac{A^2 b''^2}{(B-C)^2}\right] + \frac{1}{4}\left[A + \varepsilon B + \beta \frac{A^2 b''^2}{(B-C)^2}\right]^2 = \lambda \frac{A^4 b''^4}{(B-C)^4}$$

$$\varepsilon''^2 B^2 \frac{b''^2}{C^2} \cos^2 \alpha + b''^2 \left[\varepsilon'' + e' \frac{C^2 b^2}{(A-B)^2}\right] +$$
$$+ \frac{1}{4}\left[C + \varepsilon'' B + \beta \frac{C^2 b^2}{(A-B)^2}\right]^2 = \lambda \frac{C^4 b^4}{(A-B)^4}$$

$$\left[\varepsilon B \frac{b(B-C)}{A(A-B)} \cos \alpha - \varepsilon'' B \frac{b''(A-B)}{C(B-C)} \sin \alpha - (C-A)^2\right]^2 +$$
$$+ \beta^2 \frac{A^2 C^2 b^2 b''^2}{(B-C)^2 (A-B)^2} + \frac{1}{2}\left[A + \varepsilon B + \beta \frac{A^2 b''^2}{(B-C)^2}\right]\left[C + \varepsilon'' B + \beta \frac{C^2 b^2}{(A-B)^2}\right] -$$
$$- 2\varepsilon\varepsilon' B^2 \frac{bb''}{AC} \sin \alpha \cos \alpha + 4 b^2 b''^2 e' \frac{AC}{(B-C)(A-B)} +$$
$$+ b^2 \left[\varepsilon'' + e' \frac{C^2 b^2}{(A-B)^2}\right] + b''^2 \left[\varepsilon + e' \frac{A^2 b^2}{(B-C)^2}\right] = 6\lambda \frac{A^2 C^2 b^2 b''^2}{(B-C)^2 (A-B)^2}$$

$$\varepsilon B \frac{b}{A} \sin \alpha \left[\varepsilon B \frac{b(B-C)}{A(A-B)} \cos \alpha - \varepsilon'' B \frac{b''(A-B)}{C(B-C)} \sin \alpha - (C-A)\right] +$$
$$+ e' \frac{AC b^3 b''}{(B-C)(A-B)} + \frac{\beta}{2} \frac{ACbb''}{(B-C)(A-B)}\left[A + \varepsilon B + \beta \frac{A^2 b''^2}{(B-C)^2}\right] +$$
$$+ bb'' \left[\varepsilon + e' \frac{A^2 b'^2}{(B-C)^2}\right] = 2\lambda \frac{A^3 C b b''^3}{(B-C)^3 (A-B)} \qquad (1.27)$$

$$- \varepsilon'' B \frac{b''}{C} \cos \alpha \left[\varepsilon B \frac{b(B-C)}{A(A-B)} \cos \alpha - \varepsilon'' B \frac{b''(A-B)}{C(B-C)} \sin \alpha - (C-A)\right] +$$
$$+ e' \frac{ACbb''^3}{(B-C)(A-B)} + \frac{\beta}{2} \frac{ACbb''}{(B-C)(A-B)}\left[C + \varepsilon'' B + \beta \frac{C^2 b^2}{(A-B)^2}\right] +$$
$$+ bb'' \left[\varepsilon'' + e' \frac{C^2 b^2}{(A-B)^2}\right] = 2\lambda \frac{AC^3 b^3 b''}{(B-C)(A-B)^3} \quad \lambda = \frac{e^2}{n^4}$$

We shall substitute into equations (1.27) the quantity β, defined by the first formula of (1.24). From the substitution in (1.27) of the expressions for ϵB and $\epsilon'' B$ found from (1.25), we shall obtain a system of algebraic equations for the determination of e', λ, b, b'', $\tan a$. We will not write these equations in expanded form, since they are very unwieldy. Taking into account (1.9), we therefore have six equations relating the five aforementioned quantities. However, this system turns out to be compatible. In particular, we succeeded in establishing two solutions:

the first solution

$$\lambda = \frac{(A-B)^2 (B-C)^2}{A^4 C^4} [(A-B)(B-C) + (A+C-B)^2], \qquad (1.28)$$

$$e' = 2 \frac{(A-B)(B-C)}{A^2 C^2}, \quad b = A \sqrt{\frac{A-B}{A-C}}, \quad b'' = C \sqrt{\frac{B-C}{A-C}}, \qquad (1.29)$$

$$\operatorname{tg} \alpha = \sqrt{\frac{B-C}{A-B}} \qquad (1.30)$$

the second solution

$$\lambda = \frac{l^2}{n^4} = \frac{(C-A)(A-B)^2(C-B)^2(C-2A)^2(2C-A)^2}{H^2 A^2C^2[3AC-B(A+C)]H^2} \tag{1.31}$$

$$c' = \frac{6(A-B)(C-B)(C-A)(C-2A)^2(2C-A)^2}{H^2[3AC-B(A+C)][3AC-2B(A+C)]H^2} \tag{1.32}$$

$$b = -\frac{H}{3(C-A)}\sqrt{\frac{A-B}{C(C-2A)}}, \quad b'' = \frac{H}{3(C-A)}\sqrt{\frac{C-B}{A(2C-A)}} \tag{1.33}$$

where

$$\operatorname{tg}\alpha = \frac{2C-A}{2A-C}\sqrt{\frac{A(C-B)(2C-A)}{C(A-B)(C-A)}} \tag{1.34}$$

$$H = \sqrt{A(C-B)(2C-A)^3 + C(A-B)(C-2A)^3} \tag{1.35}$$

Substituting these expressions into (1.25) we find for the first solution

$$\varepsilon = \varepsilon'' = -1 \tag{1.36}$$

and for the second solution

$$\frac{\varepsilon}{A^2} = \frac{\varepsilon''}{C^2} = -\frac{(2C-A)(C-2A)}{[3AC-B(A+C)][3AC-2B(A+C)]} \tag{1.37}$$

Since in the formulas (1.28) to (1.37) A and C enter symmetrically, we let, without loss of generality, $C > A$.

2. Let us consider the first solution. We conclude from (1.31) and the inequality $C > A$ that

$$C > B > A, \qquad \sin\alpha = \sqrt{\frac{C-B}{C-A}}, \qquad \cos\alpha = \sqrt{\frac{B-A}{C-A}} \tag{2.1}$$

Therefore (1.29) gives

$$b = A\cos\alpha, \qquad b'' = C\sin\alpha \tag{2.2}$$

Substituting (2.2), (1.36), (1.29) into (1.11) and (1.14) we obtain, taking into account (1.30),

$$p\cos\alpha + r\sin\alpha = \nu, \quad p^2 + r^2 = 2\nu^2 - q^2, \quad \nu = -\frac{n}{AC}\sqrt{(C-B)(B-A)}$$

Hence

$$p = \sin\alpha\sqrt{\nu^2 - q^2} + \nu\cos\alpha, \qquad r = -\cos\alpha\sqrt{\nu^2 - q^2} + \nu\sin\alpha$$

Instead of q, we shall introduce the variable σ, using the equality $q = \nu \sin \sigma$. Then

$$p = \nu(\cos\alpha + \sin\alpha\cos\sigma), \qquad r = \nu(\sin\alpha - \cos\alpha\cos\sigma).$$

Formula (1.12) now gives

$$l\gamma' = \nu^2 \sin\sigma\,[A\cos^2\alpha + C\sin^2\alpha + (A-C)\sin\alpha\cos\alpha\cos\sigma]$$

We will obtain an equation for the determination of the dependence of σ on time by way of substitution of the quantities found in any of the first two equations of (1.1), taking into account (2.1):

$$\frac{d\sigma}{dt} = -\nu, \qquad \sigma = -\nu t$$

(the irrelevant integration constant is omitted).

Therefore

$$p = \nu(\cos\alpha + \sin\alpha\cos\nu t), \quad q = -\nu\sin\nu t, \quad r = \nu(\sin\alpha - \cos\alpha\cos\nu t)$$

$$l\gamma' = -\nu^2\sin\nu t\,[A\cos^2\alpha + C\sin^2\alpha + (A-C)\sin\alpha\cos\alpha\cos\nu t] \tag{2.3}$$

Substituting (1.36), (1.29), (2.2) into (1.24), we find

$$\beta = \frac{(C-B)(B-A)}{A^2C^2}(2B-A-C), \qquad \mu = \frac{\sqrt{(A-B)^3(B-C)^3}}{A^2C^2}$$

After this, having obtained from (1.20) expressions for a, a', a'', c, c', c'', and substituting these into formulas (1.19), we will find

$$l\gamma = \nu^2[C\sin\alpha\cos\nu t + (C-B)\cos\alpha\sin^2\nu t]$$

$$l\gamma'' = \nu^2[-A\cos\alpha\cos\nu t + (A-B)\sin\alpha\sin^2\nu t] \tag{2.4}$$

The cited integrability case of this Section has been obtained and investigated by Grioli [1]. This solution is given in article [2] in the form (2.3), (2.4).

3. Let us investigate the second solution. We will intoduce the following notations:

$$\cos\rho = \sqrt{\frac{C(A-B)(C-2A)}{(C-A)[3AC-B(A+C)]}}, \qquad \sin\rho = \sqrt{\frac{A(C-B)(2C-A)}{(C-A)[3AC-B(A+C)]}}$$

$$\nu = -\frac{3n}{H}\sqrt{\frac{AC(C-A)(A-B)(C-B)(C-2A)(2C-A)}{[3AC-B(A+C)]}} \tag{3.1}$$

Taking these into account, together with the substitution (1.33), the expression (1.11) will be written in the form $Ap\cos\rho + Cr\sin\rho = \nu$. The substitution of (1.32) and (1.34) into (1.11) gives

$$A^2p^2 + C^2r^2 = [3AC - B(A+C)]\left\{\frac{2\nu^2}{3AC} - q^2\frac{3AC - 2B(A+C)}{(C-2A)(2C-A)}\right\} \qquad (3.2)$$

We find from the last two relations

$$Ap = \nu\cos\rho + \sin\rho\sqrt{\frac{3AC - 2B(A+C)}{3AC}\left\{\nu^2 - \frac{3AC[3AC - B(A+C)]}{(C-2A)(2C-A)}q^2\right\}}$$

$$Cr = \nu\sin\rho - \cos\rho\sqrt{\frac{3AC - 2B(A+C)}{3AC}\left\{\nu^2 - \frac{3AC[3AC - B(A+C)]}{(C-2A)(2C-A)}q^2\right\}} \qquad (3.3)$$

Instead of q, we shall introduce the new variable σ using the equality $q = \nu\kappa\sin\sigma$, where

$$\varkappa = \sqrt{\frac{(C-2A)(2C-A)}{3AC[3AC - B(A+C)]}} \qquad (3.4)$$

Then

$$\chi = \sqrt{1 - \frac{2B(A+C)}{3AC}} \qquad (3.5)$$

Let us see what conditions must be satisfied by the positive quantities A, B, C, in order that a, n, p, r be real,

We conclude, as a consequence of $C > A$ from (1.34), that it is necessary for the inequality

$$(C-B)(A-B)(C-2A) > 0$$

to be satisfied.

This is possible when:

1) $B > C > 2A$, 2) $2A > C > B > A$, 3) $C > 2A > 2B$

In the first case

$$3AC - B(A+C) = -C(B-2A) - A(B-C) < 0$$

$$-A(B-C)(2C-A)^3 - C(B-A)(C-2A)^3 < 0$$

in the second

$$3AC - B(A+C) = C(2A-B) + A(C-B) > 0$$

$$A(C-B)(2C-A)^3 + C(B-A)(2A-C)^3 > 0$$

in the third

$$3AC - B(A+C) = A(2C-B) + C(A-B) > 0$$

$$A(C-B)(2C-A)^3 + C(A-B)(C-2A)^3 > 0$$

We see that the quantity n, determined from the equality (1.31), is real in all these cases. We conclude from (3.1), that ν will also be real.

In the first case, the multiplier within the square bracket in equation (3.2) is negative, and the second factor

$$\frac{2\nu^2}{3AC} - q^2\frac{3AC - 2B(A+C)}{(C-2A)(2C-A)} = \frac{2\nu^2}{3AC} + q^2\frac{2A(B-C) + C(2B-A)}{(C-2A)(2C-A)}$$

is positive. Therefore, the first case must be disregarded, since, under these conditions, $A^2p^2 + C^2r^2 < 0$.

In the second case, formulas (3.1) give real values for $\cos\rho$, $\sin\rho$, but the expression under the radical in (3.3) is negative. In fact,

$$[3AC - 2B(A+C)]\left\{\nu^2 - \frac{3AC\,[3AC - B(A+C)]}{(C-2A)(2C-A)}\,q^2\right\} =$$
$$= -[2C(B-A) + A(2B-C)]\left\{\nu^2 + \frac{3AC\,[3AC - B(A+C)]}{(2A-C)(2C-A)}\,q^2\right\} < 0$$

is real, since $2B > 2A > C$. In this case p and r are complex; therefore, the second case must also be disregarded.

In the third case, which is characterized by the inequality $C > 2A > 2B$, all quantities turn out to be real. Moreover $C > A + B$, which does not hold for a rigid body; however, this condition can be satisfied if the body has cavities filled with a fluid [3].

Substituting the found magnitudes into (1.12), we have

$$\gamma' = \frac{\sqrt{3AC(C-2A)(2C-A)}}{[3AC - B(A+C)]^{3/2}}\left[B - 3\chi\sqrt{\frac{AC(A-B)(C-B)}{(C-2A)(2C-A)}}\cos\sigma\right]\sin\sigma \tag{3.6}$$

The dependence of σ on time is obtained from the first equation of (1.1) in the form

$$\frac{N}{\nu}\frac{d\sigma}{dt} = k + k'\cos\sigma \tag{3.7}$$

where

$$k = \sqrt{(C-2A)(2C-A)[3AC - 2B(A+C)]}$$
$$k' = (A+C)\sqrt{3(A-B)(C-B)}$$
$$N = 3AC\sqrt{3AC - B(A+C)} \tag{3.8}$$

Having obtained from (1.24) the value of β, we next find the coefficients (1.20), after which, formulas (1.19) will give, taking into account (3.4),

$$\gamma = \frac{1}{3AC - B(A+C)}\{\cos\rho\,[3A(C-B)\sin^2\sigma + B(A+C)] + 3AC\chi\sin\rho\cos\sigma\} \tag{3.9}$$
$$\gamma'' = \frac{1}{3AC - B(A+C)}\{\sin\rho\,[3C(A-B)\sin^2\sigma + B(A+C)] - 3AC\chi\cos\rho\cos\sigma\}$$

Thus, the dependence of the variables p, q, r, γ, γ', γ'' on time is found. Hence, the Euler-Poisson equations with the conditions

$$C > 2A > 2B$$

admit a solution, determined by

$$p = \frac{\nu}{A}(\cos\rho + \chi\sin\rho\cos\sigma), \quad q = \nu\varkappa\sin\sigma, \quad r = \frac{\nu}{C}(\sin\rho - \chi\cos\rho\cos\sigma) \tag{3.10}$$

and equations (3.6) and (3.9), where $\sin\rho$, $\cos\rho$, κ, χ, are determined from (3.1), (3.4) and (3.5)

$$\nu = -3AC\sqrt{\frac{l}{H}}\sqrt[4]{\frac{C-A}{3AC - B(A+C)}}$$

The variable σ is found from equation (3.7), taking into account (3.8).

The validity of the obtained result can be verified directly by substituting the solution (3.10), (3.6), (3.9) into equations (1.1), (1.2) and the integrals (1.3), (1.4), (1.5).

Taking into account, for μ, the formula (1.24), we find from the relation $lm = \mu n^3$, that the constant of the surface integral m is equal to the constant ν, introduced according to equation (3.1), i.e. $m = \nu$.

In problems which are reducible to an integration of the Euler-Poisson equations it is customary to choose, for the main variables, Eulerian angles.

We shall introduce these angles as follows.

The formula

$$Ap\cos\rho + Cr\sin\rho = \nu \tag{3.11}$$

expresses the condition that the projection of the vector (p, q, r) on the straight line having direction of the vector

$$(A\cos\rho,\ 0,\ C\sin\rho) \tag{3.12}$$

remains constant. We will denote the angle between the vector (3.12) and the vector $(\gamma, \gamma', \gamma'')$ by θ :

$$\cos\theta = \frac{A\gamma\cos\rho + C\gamma''\sin\rho}{\sqrt{A^2\cos^2\rho + C^2\sin^2\rho}}$$

or, taking into account (3.9)

$$\cos\theta = \frac{3\,[A^2(C-B)\cos^2\rho + C^2(A-B)\sin^2\rho]\sin^2\sigma}{[3AC - B(A+C)]\sqrt{A^2\cos^2\rho + C^2\sin^2\rho}} + \frac{3AC(A-C)\chi\sin\rho\cos\rho\cos\sigma + B(A+C)(A\cos^2\rho + C\sin^2\rho)}{[3AC - B(A+C)]\sqrt{A^2\cos^2\rho + C^2\sin^2\rho}} \tag{3.13}$$

The components of the vector (p, q, r) in the directions (3.12) and $(\gamma, \gamma', \gamma'')$ are the derivatives of the angle of spin ϕ and the angle of precession ψ. The projection of (p, q, r) in the direction (3.12) is, as can be seen from (3.11, $\nu/\sqrt{A^2\cos^2\rho + C^2\sin^2\rho}$. Hence, we have

$$p\gamma + q\gamma' + r\gamma'' = \dot{\psi} + \dot{\varphi}\cos\theta, \qquad \frac{\nu}{\sqrt{A^2\cos^2\rho + C^2\sin^2\rho}} = \dot{\varphi} + \dot{\psi}\cos\theta$$

It follows that

$$\frac{d\psi}{dt} = \frac{1}{\sin^2\theta}\left(p\gamma + q\gamma' + r\gamma'' - \frac{\nu\cos\theta}{\sqrt{A^2\cos^2\rho + C^2\sin^2\rho}}\right)$$

Substituting in the last expression the values of the variables, which have already been found, we will find the dependence of $d\psi/dt$ on σ, and consequently also the dependence of the angle of precession on time.

Since the projection of (p, q, r) on the straight line having direction (3.12) is constant, it is sufficient to examine, in order to conclude the investigation, what kind of curve is described on the unit sphere with center at the fixed point by the apex, i.e. the point of intersection of this line with the sphere.

We conclude from (3.7) that when $k < k'$, the variable σ approaches asymptotically the value σ^*, determined by the relation

$$\cos\sigma^* = -k/k'$$

Moreover, the variables p, q, r, respectively, approach asymptotically the constants p^*, q^*, r^*, so that in the limit we have a permanent rotation.

For $k > k'$, σ increases indefinitely with time, and the variable θ, as can be seen from (3.11), is confined between the two limits θ_{min} and θ_{max}. Consequently, the trajectory of the apex is confined between the parallels determined by these limits for θ. We will not investigate this trajectory.

BIBLIOGRAPHY

1. Grioli, G., Esistenza e determinazione delle precesioni regolari dinamicamente possibili par solido pesante asimmetrico. *Annali Matem. pura di aplicata*, ser. 4, Vol. 26, 1947.

2. Guliaev, N.P., Ob odnom chastnom reshenii uravnenii dvizhenia tiazhelogo tverdogo tela, imeiushchego odnu nepodvizhayiu tochku (On a particular solution of the equations of motion of a heavy rigid body having a fixed point). *VMU* No.3, 1959.

3. Zhukovskii, N.E., O dvizheni tverdogo tela, imeiushchego, polosti, napolnennye odnorodnoi kapel'noi zhidkost'iu (On the motion of a heavy rigid body having cavities filled with a homogeneous dropping liquid). Complete collected works. *Polnoe sobranie sochine* Vol.3, 1936.

4. Kharlamova, E.I., Odin chastnyi sluchai integriruemosti uravnenii Eilera Puassona (A particular case of integrability of the Euler-Poisson equations). *Dokl. Akad. Nauk SSSR* Vol. 125, No. 3, 1959.

Translated by I.S.